Ferromagnets and antiferromagnets in the effective Lagrangian perspective

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Abstract. Nonrelativistic systems exhibiting collective magnetic behavior are analyzed within the framework of effective Lagrangians. The method, which formulates the dynamics of the system in terms of Goldstone bosons, allows to investigate the consequences of spontaneous symmetry breaking from a unified point of view. Analogies and differences with respect to the Lorentz-invariant situation (chiral perturbation theory) are pointed out. We then consider the low-temperature expansion of the partition function both for ferro- and antiferromagnets, where the spin waves or magnons represent the Goldstone bosons of the spontaneously broken symmetry $O(3) \to O(2)$. In particular, the low-temperature series of the staggered magnetization for antiferromagnets and the spontaneous magnetization for ferromagnets are compared with the condensed matter literature.

In condensed matter physics, spontaneous symmetry breaking is a common phenomenon and effective field theory methods are widely used in this domain. Only recently, however, has chiral perturbation theory – the effective theory of the strong interactions – been extended to such nonrelativistic systems.[1] The method applies to any system where the Goldstone bosons are the only excitations without energy gap. The essential point is that the properties of these degrees of freedom and their mutual interactions are strongly constrained by the symmetry inherent in the underlying model – the specific nature of the underlying model itself, however, is not important.

In the following presentation, our interest is devoted to the low-energy analysis of nonrelativistic systems, which exhibit collective magnetic behavior. The Heisenberg Hamiltonian is invariant under a simultaneous rotation of the spin variables, described by the symmetry group G = O(3), whereas the ground states of ferro- and antiferromagnets break this symmetry spontaneously down to H = O(2). The corresponding Goldstone modes are referred to as spin waves or magnons. Note that, in contrast to the relativistic version of Goldstone's theorem, the theorem does now neither specify the exact form of the dispersion relation at large wavelengths, nor does it determine the number of different Goldstone *particles*: these features of the Goldstone degrees of freedom are not fixed by symmetry considerations alone – rather, in the case of a Lorentz-noninvariant ground state, they depend on the specific properties of the corresponding nonrelativistic systems. Only the number of real Goldstone *fields* turns out to be universal, given by the dimension of G/H.

Indeed, it is well known that the structure of the ferromagnetic dispersion relation is quite different from the antiferromagnetic one: at large wavelengths, the former takes a *quadratic* form, whereas the latter follows a *linear* law. The mechanism which leads to this pattern and, at the same time, explains the different number of independent magnon

states – *one* for a ferromagnet, *two* for an antiferromagnet – is understood. Remarkably, in the framework of the effective description, the difference is related to the value of a single observable, the spontaneous magnetization Σ.[1, 2]

In the leading order effective Lagrangian of a *ferromagnet*, the spontaneous magnetization shows up as a coupling constant associated with a topological term involving a single time derivative,

$$\mathcal{L}^F_{eff} \,=\, \Sigma \frac{\partial_0 U^1 U^2 - \partial_0 U^2 U^1}{1+U^3} \,+\, \Sigma f^i_0 U^i \,-\, \tfrac{1}{2} F^2 D_r U^i D_r U^i . \tag{1}$$

In the above notation, the two real components of the magnon field, $U^a (a = 1,2)$ have been collected in a three-dimensional unit vector $U^i = (U^a, U^3)$. The quantity f^i_0 involves the magnetic field H: $f^i_0 = \mu H \delta^i_3$. At leading order of the low-energy expansion, the ferromagnet is thus characterized by two low-energy coupling constants, Σ and F. Note that the corresponding equation of motion (Landau-Lifshitz equation) is of the Schrödinger type: first order in time, but second order in space. As only positive frequencies occur in its Fourier decomposition, a complex field is required to describe one particle – in a ferromagnet there exists only *one* type of spin-wave excitation exhibiting a quadratic dispersion law.

The ground state of an antiferromagnet, on the other hand, does not exhibit spontaneous magnetization, such that the leading order effective Lagrangian takes the form

$$\mathcal{L}^{AF}_{eff} \,=\, \tfrac{1}{2} F^2_1 D_0 U^i D_0 U^i - \tfrac{1}{2} F^2_2 D_r U^i D_r U^i + \Sigma_s \mu h^i U^i \,, \quad D_\mu U^i = \partial_\mu U^i + \varepsilon_{ijk} f^j_\mu U^k . \tag{2}$$

Note that the anisotropy field $\vec{h}$ couples to the staggered magnetization Σ_s. The corresponding equation of motion now is of second order both in space and in time, its relativistic structure determining the number of independent magnon states: the Fourier decomposition contains both positive and negative frequencies, such that a single real field suffices to describe one particle. Accordingly, there exist *two* different types of spin-wave excitations in an antiferromagnet – as is the case in Lorentz-invariant theories, Goldstone fields and Goldstone particles are in one-to-one correspondence. These low-energy excitations follow a linear dispersion relation, with the velocity of light replaced by the spin-wave velocity $v = F_2/F_1$. As is commonly done with the velocity of light in relativistic theories, we may put the spin-wave velocity to one. In this "$\hbar = v = 1$"-system the two coupling constants F_1 and F_2 then coincide: $F_1 = F_2 \equiv \mathcal{F}$.

Accordingly, the low-energy properties of ferromagnets and antiferromagnets are quite different. As an illustration, let us consider the low-temperature expansion for the corresponding order parameters, the spontaneous and staggered magnetization, respectively.

The order parameter for an O(N) antiferromagnet, the staggered magnetization, is given by the derivative of the free energy density with respect to the anisotropy field,[3] $\Sigma_s(T) = -\partial z / \partial h$:

$$\begin{aligned} \Sigma_s(T) \,=\, \Sigma_s \Big\{ 1 \,-\, \frac{N-1}{24} \frac{T^2}{\mathcal{F}^2} \,-\, \frac{(N-1)\,(N-3)}{1152} \frac{T^4}{\mathcal{F}^4} \\ -\, \frac{(N-1)\,(N-2)}{1728} \frac{T^6}{\mathcal{F}^6} \ln \frac{T_\Sigma}{T} + O(T^8) \Big\}. \end{aligned} \tag{3}$$

The terms of order T^0, T^2, T^4 and T^6 arise from tree-, one-loop, two-loop and three-loop graphs, respectively. Up to and including T^6, the coefficients are determined by the constant $\mathcal{F}$ which thus sets the scale of the expansion. The logarithm only shows up at order T^6: the scale T_Σ involves next-to-leading order coupling constants.

Let us first consider the particular case N=4. The two groups O(4) and O(3) are locally isomorphic to SU(2) $\times$ SU(2) and SU(2), respectively. Hence, the above three-loop formula referring to the order parameter of an O(4) antiferromagnet in zero external field in fact describes the low-temperature expansion of the quark condensate of massless QCD with two flavors. This nicely illustrates the concept of *universality*: in the construction of effective Lagrangians only the mathematical structure of the groups G and H, associated with the spontaneously broken symmetry, is relevant, whereas the specific properties of the underlying model merely manifest themselves in the numerical values of the coupling constants.

Remarkably, for N=3, the T^4-term in the above formula drops out, such that we end up with the following low-temperature series for the staggered magnetization of the O(3) antiferromagnet:

$$\Sigma_s(T) \; = \; \Sigma_s \left\{ 1 \, - \, \frac{1}{12} \frac{(k_B T)^2}{\hbar v \mathcal{F}^2} \, - \, \frac{1}{864} \frac{(k_B T)^6}{\hbar^3 v^3 \mathcal{F}^6} \ln \frac{T_\Sigma}{T} + O(T^8) \right\}. \tag{4}$$

Note that we have restored the dimensions: k_B is Boltzmann's constant and v is the spin-wave velocity.

The microscopic calculation agrees with the above effective expansion up to order T^2, provided that the two coupling constants $\mathcal{F}$ and Σ_s are identified as

$$\mathcal{F}^2 \; = \; \frac{S-\sigma}{\sqrt{2z}} \frac{\hbar v}{a^2} \; = \; 2S(S-\sigma) \frac{|J|}{a} \, , \qquad \Sigma_s \; = \; \frac{g \mu_B (S-\sigma)}{a^3} \, . \tag{5}$$

The expression involves the following quantities: the exchange integral (J), the highest eigenvalue of the spin operator S^3_n (S), the number of nearest neighbors of a given lattice site (z), the length of the unit cell (a), the Landé factor (g), the "Anderson factor" (σ) and the Bohr magneton (μ_B). Note that the spin-wave velocity v is given by the following combination of microscopic quantities,

$$v = 2|J| S \sqrt{2z} \, a / \hbar . \tag{6}$$

The scale of the low-temperature expansion is set by $\mathcal{F} \sqrt{\hbar v}$ – let us briefly estimate its value. Written in terms of the exchange integral J, we obtain

$$\mathcal{F} \sqrt{\hbar v} = 2|J| S \sqrt{(S-\sigma)\sqrt{2z}} . \tag{7}$$

Now, for a simple cubic lattice ($z = 6, \sigma = 0.078$) and for $S = 1/2$, the double square root on the right hand side is approximately equal to one, such that we end up with $\mathcal{F} \sqrt{\hbar v} \approx |J|$. Typically, the exchange integral for antiferromagnets is around $|J| \approx 10^{-3} eV$, and the scale $\mathcal{F} \sqrt{\hbar v}$ thus of the same order of magnitude. This is to be contrasted with the situation in QCD, where the relevant quantity, $F_\chi \sqrt{\hbar c}$, takes the value $92 MeV$ – the respective scales in the two theories thus differ in about eleven orders of magnitude.

As far as subleading terms in the expansion of the staggered magnetization are concerned, it is well known that a T^4-contribution is absent: the spin-wave interaction only manifests itself at higher orders. However, the logarithmic dependence on the temperature is not found in a microscopic calculation. We conclude that it is extremely difficult to calculate the corrections of order T^6 in the framework of a microscopic theory.

Let us now turn to the ferromagnet. The low-temperature expansion for the spontaneous magnetization is given by the derivative of the free energy density with respect to the magnetic field and takes the form[4]

$$\Sigma(T)/\Sigma \, = \, 1 - \alpha_0 T^{\frac{3}{2}} - \alpha_1 T^{\frac{5}{2}} - \alpha_2 T^{\frac{7}{2}} - \alpha_3 T^4 + O(T^{\frac{9}{2}}). \tag{8}$$

The coefficients α_n are independent of the temperature and involve the various coupling constants occurring in the effective Lagrangian, which phenomenologically parametrize the microscopic detail of the system.

In the above series, half-integer powers of the temperature correspond to *noninteracting* magnons; these contributions can be absorbed into a redefinition of the dispersion relation. Remarkably, the leading term describing the magnon-magnon *interaction* (two-loop graph) is of order T^4, and clearly confirms Dyson's microscopic calculation.[5] The effective Lagrangian technique, however, proves to be more efficient than conventional condensed matter methods as the analysis can be carried to higher orders: the calculation shows that the next interaction term, which arises at the three-loop level, is of order $T^{9/2}$.

The effective Lagrangian method is also more transparent, since it addresses the problem from a model-independent point of view based on symmetry – at large wavelengths, the microscopic structure of the system only manifests itself in the numerical values of a few coupling constants.

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